FOURIER TRANSFORM AND THE VIBRATING STRING

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The solution of the vibrating string problem using Fourier series applied to a string of finite length with its initial position and velocity specified. We can extend this to the case of an infinite string (OK, such a string doesn't actually exist, but anyway...). The motion of the string is given by the wave equation

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial^2 u(x,t)}{\partial t^2} \tag{1}$$

The initial conditions are given by

$$\begin{aligned} u(x,0) &= f(x) \\ \frac{\partial u}{\partial t}(x,0) &= 0 \end{aligned} \tag{2}$$

That is, the string's initial shape is given by some function $f(x)$, and its initial velocity is zero everywhere.

A solution is provided by

$$u(x,t) = \int_{-\infty}^{\infty} G(\omega) e^{i\omega x} \cos \omega t \, d\omega \tag{3}$$

where $G(\omega)$ is the Fourier transform of $f(x)$:

$$G(\omega) = \int_{-\infty}^{\infty} f(\xi) e^{-i\omega\xi} d\xi \tag{4}$$

with the inverse transform given by

$$\int_{-\infty}^{\infty} G(\omega) e^{i\omega x} d\omega = f(x) \tag{5}$$

This satisfies the wave equation 1 (assuming we can differentiate under the integral sign), since

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \int_{-\infty}^{\infty} G(\omega)(i\omega)^2 e^{i\omega x} \cos \omega t \, d\omega = -\int_{-\infty}^{\infty} G(\omega) \omega^2 e^{i\omega x} \cos \omega t \, d\omega \tag{6}$$

$$\frac{\partial^2 u(x,t)}{\partial t^2} = -\int_{-\infty}^{\infty} G(\omega)\,\omega^2 e^{i\omega x}\cos\omega t\, d\omega = \frac{\partial^2 u(x,t)}{\partial x^2} \tag{7}$$

The initial conditions are also satisfied:

$$u(x,0) = \int_{-\infty}^{\infty} G(\omega)\, e^{i\omega x}\cos 0\, d\omega \tag{8}$$

$$= \int_{-\infty}^{\infty} G(\omega)\, e^{i\omega x}\, d\omega \tag{9}$$

$$= f(x) \tag{10}$$

and

$$\frac{\partial u}{\partial t}(x,0) = -\int_{-\infty}^{\infty} G(\omega)\, e^{i\omega x}\omega\sin 0\, d\omega = 0 \tag{11}$$

If we swap the initial conditions so that

$$\begin{aligned} u(x,0) &= 0 \\ \frac{\partial u}{\partial t}(x,0) &= g(x) \end{aligned} \tag{12}$$

for some given function $g(x)$, we can find the Fourier transform of $g(x)$ as

$$H(\omega) = \int_{-\infty}^{\infty} g(\xi)\, e^{-i\omega\xi} d\xi \tag{13}$$

with inverse transform

$$\int_{-\infty}^{\infty} H(\omega)\, e^{i\omega x} d\omega = g(x) \tag{14}$$

We try a solution of form

$$u(x,t) = \int_{-\infty}^{\infty} K(\omega)\, e^{i\omega x}\sin\omega t\, d\omega \tag{15}$$

where $K(\omega)$ is a function to be determined.

To check the initial conditions 12 we have

$$u(x,0) = \int_{-\infty}^{\infty} K(\omega)\, e^{i\omega x}\sin\omega t\, d\omega \tag{16}$$

$$= \int_{-\infty}^{\infty} K(\omega)\, e^{i\omega x}\sin 0\, d\omega \tag{17}$$

$$= 0 \tag{18}$$

Also

$$\frac{\partial u}{\partial t}(x,0) = \int_{-\infty}^{\infty} \omega K(\omega)\, e^{i\omega x}\cos 0\, d\omega \tag{19}$$

$$= \int_{-\infty}^{\infty} \omega K(\omega)\, e^{i\omega x}\, d\omega \tag{20}$$

In order that this is equal to $g(x)$ we have, from 14

$$\omega K(\omega) = H(\omega) \tag{21}$$

so

$$u(x,t) = \int_{-\infty}^{\infty} \frac{H(\omega)}{\omega} e^{i\omega x} \sin \omega t \, d\omega \tag{22}$$

As with the finite string, we can combine these two results to get the solution for a string with general initial conditions

$$u(x,0) = f(x) \tag{23}$$

$$\frac{\partial u}{\partial t}(x,0) = g(x) \tag{24}$$

The result is

$$u(x,t) = \int_{-\infty}^{\infty} G(\omega) e^{i\omega x} \cos \omega t \, d\omega + \int_{-\infty}^{\infty} \frac{H(\omega)}{\omega} e^{i\omega x} \sin \omega t \, d\omega \tag{25}$$

As an example, we set

$$\begin{aligned} f(x) &= e^{-x^2} \\ g(x) &= 0 \end{aligned} \tag{26}$$

The Fourier transform of $f(x)$ is

$$G(\omega) = \frac{e^{-\omega^2/4}}{2\sqrt{\pi}} \tag{27}$$

so the motion of the string is given by

$$u(x,t) = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\omega^2/4} e^{i\omega x} \cos \omega t \, d\omega \tag{28}$$

Surprisingly, this integral has an exact form (given by Maple; please don't ask me how you get it!):

$$u(x,t) = 2\sqrt{\pi} e^{-t^2 - x^2} \cosh(2tx) \tag{29}$$

$$= \sqrt{\pi} e^{-t^2 - x^2} \left(e^{2tx} + e^{-2tx}\right) \tag{30}$$

$$= \sqrt{\pi} \left(e^{-(t-x)^2} + e^{-(t+x)^2}\right) \tag{31}$$

A plot of this is shown in Fig. 1 for $x \in [-10, 10]$ and time $t \in [0, 10]$. The initial displacement around $x = 0$ gives rise to two waves that propagate outwards from $x = 0$, along the lines $x = \pm t$. The amplitude of the waves rapidly approaches $\sqrt{\pi} \approx 1.77$.

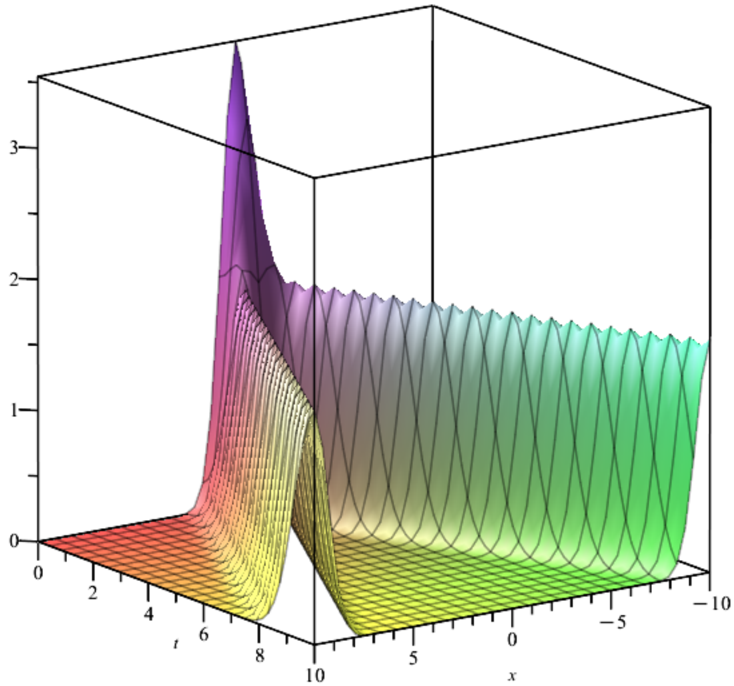

FIGURE 1. Plot of $u(x,t) = 2\sqrt{\pi}e^{-t^2-x^2}\cosh(2tx)$.